Projective modules over the kernel of a locally nilpotent derivation on a polynomial ring

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Theorem (Quillen-Suslin):
All projective modules over the polynomial ring $k[X_1, \ldots, X_n]$ over a field k are necessarily free.
(Was known as Serre's conjecture for 20 years.)

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A k-derivation D of B is a k-linear map

$$D : B \to B \text{ such that}$$

$$D(ab) = aD(b) + bD(a) \;\; \forall \;\; a, b \in B.$$

D is called **locally nilpotent derivation** (LND) if, for every $a \in B$, $\exists\ n \geq 1$ (depending on a) such that $D^n(a) = 0$.

Let D be a non-zero LND on the affine k-domain B with kernel A, i.e.,

$$A := \mathrm{Ker}(D) = \{a \in B \mid D(a) = 0\} \subseteq B.$$

- $D(\lambda) = 0 \ \forall \ \lambda \in k$, i.e., $k \subseteq \mathrm{Ker}(D)$.

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- $A^* = B^*$.
- dim $A =$ dim $B - 1$.
- A is factorially closed in B, i.e., if $a \in A$ such that $a = bc$ for some $b, c \in B$, then $b, c \in A$.

Let D be a non-zero LND on the affine k-domain B. For any $\lambda \in k$, the map

$$\exp(\lambda D) : B \to B \text{ specifying } b \longmapsto \sum_{i=0}^{\infty} \frac{D^i(b)}{i!} \lambda^i \ (b \in B)$$

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LND useful in knowing the structure of $\mathrm{Aut}_k(B)$.

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Such derivations are called *triangular derivations*.

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- (Rentschler 1968) When $n = 2$ (i.e., $B_2 = k[X_1, X_2]$), $A_1 = k[F]$ and $B_2 = k[F, G]$ for some $G \in B_2$.

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- (Miyanishi 1983) When $n = 3$ (i.e., $B_3 = k[X_1, X_2, X_3]$), $A_2 = k[f, g]$. (However, B_3 need not be of the form $A_2[H]$.)

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- G. Freudenburg (1998) constructed an interesting LND D on $B_3 = k[X_1, X_2, X_3]$ such that A_2 does not contain any variable of B_3.

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- For $n = 4$, it is not known whether the kernel of any LND on B_4 is always a f.g. k-algebra.

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Though it can be bad enough, the kernel A_{n-1} inherits some of the properties (like factoriality) of the polynomial ring B_n and it is expected that A_{n-1} enjoys, at least to some extent, several other properties of B_n. In that spirit, Miyanishi asked:

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What if $n = 4$?

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Investigate the question of Miyanishi for the case $n = 4$ and under the additional hypothesis that D annihilates a variable of B. For convenience, assume that $D(X_1) = 0$.

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There exists D such that A has (only) isolated singularities but projective modules over A are not necessarily free.

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We may also take $D(U) = (X - a_1) \cdots (X - a_m)$, $a_i \neq a_j$.

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$$K_0(R) := \frac{\bigoplus_{P \in \mathcal{H}} \mathbb{Z}.P}{\langle P_2 - P_1 - P_3 \mid 0 \to P_1 \to P_2 \to P_3 \to 0 \rangle}.$$

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For any f.g. R-module P, let $[P]$ denote its image in $K_0(R)$.

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By Quillen Suslin Theorem, $K_0(k[X_1, \ldots, X_n]) = \mathbb{Z}$.

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1. *C is a non-singular (regular) affine rational curve.*
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Thus, if C is not a regular ring then $K_0(E)$ is not finitely generated; in particular, there exists an infinite family of non-isomorphic projective E-modules which are not even stably isomorphic.

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Question: Is the converse also true?

Answer: Yes, i.e., TFAE:

- $E \cong \mathrm{Ker}(D)$ for some triangular LND D of $k[X_1, X_2, X_3, X_4]$.
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In fact, $G_1(A) = G_1(B) = k^*$.

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Question: *Suppose that all projective A-modules are free. Does it follow that A is a regular ring?, i.e., Does it follow that* $A \cong k[X, Y, Z]$*?*